Sorbonne Université M2 - B004

Méthodes numériques pour les EDP instationnaires

TP 3: jeudi 14.10.2021
Error measurements

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1 Error measurements

This is a description of the **standard method** to perform measurements of numerical errors. You can use the file `https://www.ljll.math.upmc.fr/despres/BD_fichiers/calcul_error.py` if you want.

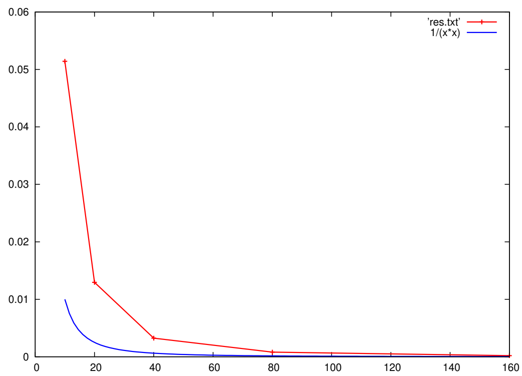
1. Implements the upwind scheme
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0.$$
2. Take a smooth initial data and write the exact analytical solution at time $t = 1$.
3. For a number cell N =10, 20, 40, 80, 160, 320, 640 compute the numerical solution at final time and measure the error between the exact solution and the numerical solution in the L^∞ norm.

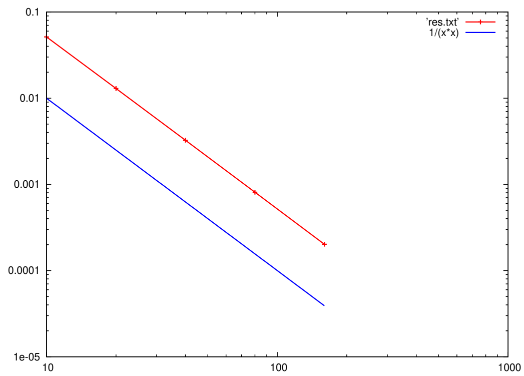
   Write the result in a file `res.txt` where the first column is N and the second column is the error.
4. Plot the results in loglog plot.
   With gnuplot:
```
gnuplot
plot res.txt
set logscale
rep
```
   or with matplotlib:
```
import matplotlib.pyplot as plt
plt.loglog(x,y)
```
   You must observe approximatively a straight line.
5. Compare the slope with the theoretical order of convergence: you must get a slope $p = 1$.
6. If the initial data is discontinuous, observe $p = 0$. Explain.
7. Show with numerical experiments that in this case, $p = \frac{1}{2}$ in the L^1 norm and $p = \frac{1}{4}$ in the L^2 norm.
8. Do the same for the Lax-Wendroff scheme.
9. Do the same for the 3 points scheme for the heat equation.

The following example is for the heat equation. One gets (for a convenient exact solution, typically a cosine)

cells	10	20	40	80	160
L^2-error	0.051417	0.012956	0.003247	0.000811	0.000202

This table is represented graphically in the figures below: on the left with usual scales; on the right with logarithmic scales. Additionally the function $x \mapsto x^2$ is displayed to evidence the slope which is the order $p = 2$.





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2 First steps with the Burgers equation

The Burgers equation in the domain $x \in \mathbb{R}$ writes

$$\partial_t u + \partial_x \frac{u^2}{2} = 0, \quad t > 0,$$

with the initial condition $u(x, 0) = u_0(x)$. For the simplicity, we assume the initial data is C^1_x and non negative

$$u_0(x) \geq 0 \text{ for all } x.$$

The Burgers equation is **non-linear**. It admits the **quasi-linear** form

$$\partial_t u + u\partial_x u = 0, \quad t > 0,$$

provided the solution is smooth enough (C^1_{xt} is enough of course).

1. Take $u_0(x) = 1$ for $x < -0.5$, $u_0(x) = 0.5 - x$ for $-0.5 < x < 0.5$ and $u_0(x) = 0$ for $0.5 < x$. Solve the quasi-linear form with the method of characteristics: $u(x, t) = u_0(X)$ where one has

$$x = X + tu_0(X). \tag{1}$$

   In particular calculate $u(x, T)$ for $T = 1$ (this is the creation of a shock).

   What can be said concerning the regularity of the solution?

2. For a numerical confirmation, upload the script `https://www.ljll.math.upmc.fr/despres/BD_fichiers/burgers.py` where the explicit scheme is implemented

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{(u_j^n)^2 - (u_{j-1}^n)^2}{2\Delta x} = 0.$$

   Check the scheme satisfies $u_j^n \geq 0$ (for all j and n) under CFL, then check the maximum principle.

3. Compute the numerical solution for $T = 1$ and $T = 1.5$, and comment.

4. Solve the equation (1) for $u_0 \geq 0$ but $u_0' \leq 0$ and show no shock can occur.
For numerical confirmation, take $u_0(x) = 0$ for $x < -0.5$, $u_0(x) = x + 0.5$ for $-0.5 < x < 0.5$ and $u_0(x) = 1$ for $0.5 < x$.

5. Open question.
Assume $u_0(x) = 1$ for $x < 0$ and $u_0(x) = 0$ for $0 < x$. Prove that the speed of the discontinuity is $\sigma = \frac{1}{2}$.