Méthodes numériques pour les EDP instationnaires

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Stability/consistency and transport equation with variable coefficients

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1 Stability and consistency via Fourier

For the heat equation

$$\begin{cases} \partial_t u(x,t) = \partial^2_{xx} u(x,t), & \forall\,(x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x), & \forall\, x \in \mathbb{R}, \end{cases} \tag{1}$$

we consider two schemes:

- an explicit 3-point scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \quad n \in \mathbb{N},\ j \in \mathbb{Z},$$

- a Dufort-Frankl type scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^{n+1} + u_{j-1}^n}{\Delta x^2}, \quad n \in \mathbb{N},\ j \in \mathbb{Z}.$$

We introduce $\nu = \frac{\Delta t}{\Delta x^2}$.
Q1: Write the symbol of the operator, the discrete symbol and study the consistency and the convergence of the explicit 3-point scheme.
Q2: Write the discrete symbol of the Dufort-Frankl type scheme and study its consistency.

For the transport equation,

$$\begin{cases} \partial_t u(x,t) + a\partial_x u(x,t) = 0, & \forall\,(x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x), & \forall\, x \in \mathbb{R}, \end{cases} \tag{2}$$

Q3: Show that the Fourier symbol of Lax-Friedrichs scheme

$$\frac{2u_j^{n+1} - u_{j+1}^n - u_{j-1}^n}{2\Delta t} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \;=\; 0$$

is not consistent (for small Δt).
Q4: Show that the Beam-Warming scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{3u_j^n - 4u_{j-1}^n + u_{j-2}^n}{2\Delta x} - \frac{a^2\Delta t}{2}\frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{\Delta x^2} = 0\,.$$

is L^2-stable for $\nu = \frac{a\Delta t}{\Delta x} \leq 2$ and second order in time and space.

2 Transport equation with variable velocity

Take a function $a \in C^1(\mathbb{R})$ such that there exist $A, A_1, A_2 \in [0,+\infty)$ with $|a(x)| \leq A$, $|a'(x)| \leq A_1$ and $|a''(x)| \leq A_2$ for all $x \in \mathbb{R}$. The nonconservative transport equation is

$$\begin{aligned} \partial_t \bar{u}(x,t) + a(x)\;\partial_x \bar{u}(x,t) = 0, &\qquad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+_*, \\ \bar{u}(x,0) = u_0(x), &\qquad \forall x \in \mathbb{R}. \end{aligned} \tag{3}$$

The conservative transport equation is

$$\begin{aligned} \partial_t \hat{u}(x,t) + \partial_x(a(x)\hat{u}(x,t)) &= 0, \qquad && \forall (x,t) \in \mathbb{R}\times\mathbb{R}_*^+, \\ \hat{u}(x,0) &= u_0(x), && \forall x \in \mathbb{R}. \end{aligned} \tag{4}$$

We assume that $u_0 \in C^2(\mathbb{R})$ with bounded derivatives. We introduce a discretization of the domain using a regular mesh:

$$(x_j, t_n) = (j\Delta x, n\Delta t), \qquad \forall j \in \mathbb{Z},\ \forall n \in \mathbb{N}\,,$$

where Δx, respectively Δt, denotes the space step, respectively the time step.
We denote $a_j = a(x_j)$, $a_j^+ = \max(a_j, 0)$, $a_j^- = \max(-a_j, 0)$. Note the relation $a_j = a_j^+ - a_j^-$.

2.1 Scheme for equation (3): convergence with Lax approach

The scheme is

$$\Delta x \frac{u_j^{n+1} - u_j^n}{\Delta t} + a_j^-(u_j^n - u_{j+1}^n) - a_{j-1}^+(u_{j-1}^n - u_j^n) = 0\,. \tag{5}$$

1. Define the discrete iteration operator $J_{h,\Delta t}$ such that $U^{n+1} = J_{h,\Delta t}U^n$.
2. Check that under a CFL condition, the scheme satisfies the discrete maximum principle and thus deduce the L^∞ stability of the scheme.
3. We assume that the solution $\bar{u}$ of (3) satisfies $\bar{u} \in C^2(\mathbb{R}\times\mathbb{R}_+)$, where the absolute value of all first and second order derivatives of $\bar{u}$ are bounded by some value C_2, and we consider the truncation error

$$T_j^n = \frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} + a_j^- \frac{\bar{u}_j^n - \bar{u}_{j+1}^n}{\Delta x} - a_{j-1}^+ \frac{\bar{u}_{j-1}^n - \bar{u}_j^n}{\Delta x}\,,$$

   where $\bar{u}_j^n = \bar{u}(x_j, t_n)$. Prove that $|T_j^n| \le C_3(\Delta t + \Delta x)$.
4. Prove the convergence in $L^\infty(\mathbb{R})$ for $u_0 \in W^{2,\infty}(\mathbb{R})$.

2.2 Scheme for equation (4): convergence with Lax approach

The scheme is

$$\Delta x \frac{u_j^{n+1} - u_j^n}{\Delta t} + (a_j^+ u_j^n - a_j^- u_{j+1}^n) - (a_{j-1}^+ u_{j-1}^n - a_{j-1}^- u_j^n) = 0\,. \tag{6}$$

1. Define the discrete iteration operator $J_{h,\Delta t}$ such that $U^{n+1} = J_{h,\Delta t}U^n$.
2. Assume $u_j^0 \geqslant 0$ for all $j \in \mathbb{Z}$. Find a CFL condition such that $u_j^n \geqslant 0$ for all $j \in \mathbb{Z}$ and all $n \in \mathbb{N}$.
3. Assume $\sum_{j\in\mathbb{Z}} |u_j^0| < \infty$. Under the same CFL condition, prove that $\sum_{j\in\mathbb{Z}} |u_j^{n+1}| < \infty$ for all $n \in \mathbb{N}$.
4. Under the previous assumptions, prove that the scheme preserves the discrete mass, *i.e* for a given $n \in \mathbb{N}$,

$$\sum_{j\in\mathbb{Z}} \Delta x u_j^{n+1} = \sum_{j\in\mathbb{Z}} \Delta x u_j^n\,.$$

5. Study the L^1 stability of the scheme.
6. Prove the convergence in $L^1(\mathbb{R})$ for compactly supported initial data.

3 Mixed cases

Mixed schemes are schemes which are stable in some norm (typically L^2) but not in L^∞.
The Crank-Nicholson (second order in time) discretization of the heat equation is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+\frac{1}{2}} - 2u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}}{\Delta x^2}\,, \qquad\qquad u_j^{n+\frac{1}{2}} = \frac{1}{2}(u_j^n + u_j^{n+1})\,.$$

1. Show it is unconditionally stable in L^2.
2. Show it is conditionally unitary stable in L^∞.

We note that it is possible to prove that

$$\|J_{h,\Delta t}\|_{\mathcal{L}(L^\infty)} = \begin{cases} 1\,, & 0 < \nu \le \frac{3}{2}\,, \\ 3 - \frac{4}{\sqrt{1+2\nu}}\,, & \nu \ge \frac{3}{2}\,. \end{cases}$$

See for example Farago & Palenciab, *Sharpening the estimate of the stability constant in the maximum-norm of the Crank–Nicolson scheme for the one-dimensional heat equation*, Applied Numerical Mathematics, 2002.

4 Transport on nonuniform meshes

Consider a nonuniform mesh with mesh sizes $0 < \alpha h < \Delta x_j \le h$. The Finite Volume scheme for advection $a > 0$ writes

$$\Delta x_j \frac{u_j^{n+1} - u_j^n}{\Delta t} + a u_j^n - a u_{j-1}^n = 0, \qquad j \in \mathbb{Z}, \quad n \in \mathbb{N}\,.$$

1. Write the scheme under the form $\frac{u_h^{n+1} - u_h^n}{\Delta t} = A_h u_h^n$.
2. Study the stability in L^∞ of the iteration operator $J_{h,\Delta t} = I_h + \Delta t A_h$.
3. For $x_j = \frac{1}{2}(x_{j+\frac{1}{2}} + x_{j-\frac{1}{2}})$ the middle of cell number j, define the interpolation of the exact solution as $v_h^n = (u(x_j, t_n))_{j \in \mathbb{Z}}$. Show this approach does not yield consistency.
4. Define another interpolation of the exact solution $w_h^n = \Big(u(x_{j+\frac{1}{2}}, t_n)\Big)_{j\in\mathbb{Z}}$. Show consistency and convergence.
5. Show stability in all l^p equipped with the norm $||v_h||_p = \Big(\sum_j \Delta x_j |v_j|^p\Big)^{\frac{1}{p}}$.

5 Compactness techniques

Compactness techniques, when applied to numerical analysis, often provide a different strategy to prove convergence, however without an explicit calculation of the rate of convergence. It can be generalized to nonlinear equations and numerical schemes as well (a main asset, but not detailed in this course). Usually the proof has three steps: (a) an estimate of the discrete derivative; (b) a compactness result; (c) convergence to a solution.

Here we focus on BV (Bounded Variation) techniques. The characterization of the BV norm that we consider is

$$BV(\mathbb{R}^d) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^d),\ |u|_{BV} := \sup_{\varphi \in W^{1,\infty}_{b,0}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u(x) \nabla \cdot \varphi(x) \mathrm{d}x < \infty \right\},$$

where

$$W^{1,\infty}_{b,0}(\mathbb{R}^d) = \left\{ \varphi \in \Big(W^{1,\infty}_0(\mathbb{R}^d)\Big)^d : \|\varphi\|_{L^\infty(\mathbb{R}^d)} \le 1 \right\}$$

is the space of compactly supported vector fields in $W^{1,\infty}(\mathbb{R}^d)$ bounded by one.

One has the Helly's selection theorem: if $(u_n)_n$ is a bounded sequence in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then there exists $u \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ such that, up to the extraction of a subsequence, $\lim_{n\to\infty} \|u_n - u\|_{L^1_{\mathrm{loc}}(\mathbb{R})} = 0$.
In the context of numerical methods, see Godlewski-Raviart, *Hyperbolic systems of conservations laws*, Ellipse, 1991, page 53.
As similar compactness result holds for functions in $L^1(\Omega) \cap BV(\Omega)$ provided $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary (Giusti, *Minimal surfaces and functions with bounded variations*, 1984).

(0) Properties of $BV(\mathbb{R})$

1. For $u \in W^{1,1}(\mathbb{R})$, show that $|u|_{BV(\mathbb{R})} = \|u'\|_{L^1(\mathbb{R})}$.
2. Let $u(x) = 1$ for $-1 < x < 1$ and $u(x) = 0$ otherwise. Show that $|u|_{BV(\mathbb{R})} = 2$.
3. Let $u_h = (u_j)_{j\in\mathbb{Z}} \in L^1(\mathbb{R})$ be a numerical profile, that is
$$u_h(x) = u_j \text{ for } (j - \frac{1}{2})h < x < (j + \frac{1}{2})h.$$
Show that $|u|_{BV(\mathbb{R})} = \sum_{j\in\mathbb{Z}} |u_j - u_{j-1}|$.

(a) Discrete estimates Now we apply this material to the numerical scheme.

1. Take the scheme (3). Show that, under CFL, it can be recast under the Harten form
$$u_j^{n+1} = (1 - C_j - D_j)\, u_j^n + C_j u_{j-1}^n + D_j u_{j+1}^n$$
with $0 \le C_j, D_j$ and $C_j + D_j \le 1$.
2. Show that, under CFL, $\sum_{j\in\mathbb{Z}} \left|u_j^{n+1} - u_{j-1}^{n+1}\right| \le \sum_{j\in\mathbb{Z}} \left|u_j^n - u_{j-1}^n\right|$.
3. Assume that the initial data $u_0 \in BV(\mathbb{R})$, show that, under CFL, a 2D discrete BV inequality holds
$$\sum_{0\le n\le T/\Delta t} \left(\sum_{j\in\mathbb{Z}} \frac{\left|u_j^n - u_{j-1}^n\right|}{\Delta x} + \sum_{j\in\mathbb{Z}} \frac{\left|u_j^{n+1} - u_j^n\right|}{\Delta t} \right) \Delta x \Delta t \le C|u_0|_{BV(\mathbb{R})}. \tag{7}$$

(b) Compactness result Defining $u_h(t,x) = u_j^n$ for $(x,t) \in \Omega_j^n$, where
$$\Omega_j^n := \{(x,t) \mid (j - 1/2)h < x < (j + 1/2)h\} \text{ and } n\Delta t < t < (n+1)\Delta t,$$
the left hand side of (7) is equivalent to $|u_h|_{BV([0,T]\times\mathbb{R})}$, so we have $|u_h|_{BV([0,T]\times\mathbb{R})} \le C|u_0|_{BV(\mathbb{R})}$.
By compactness, there exists a subsequence converging to u in $L^1_{\text{loc}}([0,T]\times\mathbb{R})$ for $h = \Delta x \to 0$ with Δt fixed such that the CFL is satisfied.
Note that slightly different but perhaps more intuitive compactness result can be used: Leveque, *Numerical Methods for Conservation Laws*, 1990, page 162.

(c) Convergence to a solution The aim is now to prove that the extracted u is a solution of the original equation. To simplify the notation, we consider that $a(x) = a > 0$ is a constant.

1. Show the following identity for all $\varphi \in C_0^1([0,\infty)\times\mathbb{R})$:
$$\sum_{0\le n\le T/\Delta t} \sum_{j\in\mathbb{Z}} \Delta x \Delta t \left(\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a(u_j^n - u_{j-1}^n)}{\Delta x} \right) \varphi(x_j, t_n) = 0.$$
2. Recast as
$$\sum_{1\le n\le T/\Delta t} \sum_{j\in\mathbb{Z}} \Delta x \Delta t u_j^n \left(\frac{\varphi(x_j, t_{n-1}) - \varphi(x_j, t_n)}{\Delta t} \right) - \sum_{j\in\mathbb{Z}} \Delta x u_j^0 \varphi(x_j, 0)$$
$$+ \sum_{0\le n\le T/\Delta t} \sum_{j\in\mathbb{Z}} \Delta x \Delta t u_j^n \left(\frac{a\left(\varphi(x_j, t_n) - \varphi(x_{j+1}, t_n)\right)}{\Delta x} \right) = 0.$$
3. Show that each part admits a limit (up to subsequence extraction), and that the limit satisfies
$$-\int_0^T \int_{\mathbb{R}} u(x,t) \left(\partial_t \varphi(x,t) + a\partial_x \varphi(x,t)\right) \mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}} u_0(x)\varphi(x,0)\mathrm{d}x = 0 \tag{8}$$
for all $\varphi \in C_0^1([0,\infty)\times\mathbb{R})$.
4. Assume some extra regularity, for example $u \in C^1([0,T]\times\mathbb{R})$. Show that u is the unique solution to $\partial_t u + a\partial_x u = 0$ plus initial condition.
5. Assume much less regularity $u \in L^1_{\text{loc}}([0,T]\times\mathbb{R})$. Show that (8) yields that $u(x,t) = u_0(x - at)$ for almost all (x,t).