Méthodes numériques pour les EDP instationnaires

TD 2: vendredi 22.09.2023
Stability and Fourier analysis

Exercise 1: Finite volume schemes

A finite volume scheme for

$$\begin{cases} \partial_t \bar{u} + a\ \partial_x \bar{u} = 0\,, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+_*, \\ \bar{u}(x,0) = u_0(x)\,, & \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

can be written

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\ \frac{f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}}}{\Delta x} \ = \ 0, \tag{2}$$

where $f^n_{j\pm\frac{1}{2}}$ denotes a numerical flux. We still denote $\nu = \frac{a\Delta t}{\Delta x}$.
We consider the following schemes

- *Upwind scheme*

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + a\ \dfrac{u_j^n - u_{j-1}^n}{\Delta x} \ = \ 0\ \text{, if } a > 0, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + a\ \dfrac{u_{j+1}^n - u_j^n}{\Delta x} \ = \ 0\ \text{, if } a < 0. \end{cases}$$

- *Lax-Friedrichs*

$$\frac{2u_j^{n+1} - u_{j+1}^n - u_{j-1}^n}{2\Delta t} + a\ \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \ = \ 0.$$

- *Lax-Wendroff*

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{a^2\Delta t}{2}\frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{\Delta x^2} = 0\,.$$

1. Check that the *upwind*, *Lax-Friedrichs* and *Lax-Wendroff* schemes can be seen as a finite volume scheme

Exercise 2: Modified equations

The following schemes are used to approximate the solution to the advection equation

$$\begin{cases} \partial_t \bar{u} + a\ \partial_x \bar{u} = 0\,, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+_*, \\ \bar{u}(x,0) = u_0(x)\,, & \forall x \in \mathbb{R}, \end{cases}$$

1. (a) Determine the modified equation of the Lax-Wendroff scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{a^2\Delta t}{2}\frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{\Delta x^2} = 0\,.$$

   (b) Suggest a modification to the Lax-Wendroff scheme such that the resulting scheme is of higher order.

2. Do the same for the Beam-Warming scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{3u_j^n - 4u_{j-1}^n + u_{j-2}^n}{2\Delta x} - \frac{a^2\Delta t}{2}\frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{\Delta x^2} = 0\,.$$

Exercise 3: Maximum principle for advection

Show that a second order (space-time) linear explicit scheme cannot preserve the maximum principle for all CFL number.

Hint: Show that second order polynomials are exactly evolved with such a scheme.

Exercise 4: introduction to the Hille-Yoshida theorem

This exercise can be considered as a pedestrian introduction to a much broader topic, see for example the Chapter 7 on the Hille-Yoshida theorem of the book *Functional Analysis, Sobolev Spaces and Partial Differential Equations* by H. Brezis. The aim is to give a meaning to the representation formula $u(t) = e^{At}u_0$ where A is an unbounded operator. Typically $A = -a\partial_x$ or $A = \partial_{xx}$. The issue is that

$$e^{At} = \sum_{n\geq 0} A^n t^n$$

is meaningless for unbounded operators. One needs to construct such an object by an approximation procedure.

1. We consider the Fourier decomposition $v(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{v}(\theta)e^{i\theta x}d\theta$ and the truncation in Fourier space

   $$v_\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} \hat{v}(\theta)e^{i\theta x}d\theta, \quad \lambda > 0.$$

   Show that $\lim_{\lambda\to\infty} v_\lambda = v$ for $v \in L^2(\mathbb{R})$. *Hint:* use the Plancherel identity.

   Measure the difference in $L^2(\mathbb{R})$ for $v \in H^s(\mathbb{R})$, $s \geq 1$.

2. Define the space

   $$X_\lambda = \left\{v \in L^2(\mathbb{R}),\ \widehat{v}(\theta) = 0 \text{ for } |\theta| > \lambda\right\}.$$

   Show that

   $$\|a\partial_x\|_{\mathcal{L}(X_\lambda)} \leq a\lambda.$$

3. Give a meaning in $\mathcal{L}(X_\lambda)$ to the series

   $$e^{-a\partial_x t} = \sum_{n\geq 0} \frac{1}{n!}(-a\partial_x t)^n.$$

4. Consider the problem ($a > 0$)

   $$\begin{cases} \partial_t u_\lambda + a\partial_x u_\lambda = 0, & t > 0, \quad x \in \mathbb{R}, \\ u_\lambda(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} \hat{u}_0(\theta)e^{i\theta x}d\theta & \end{cases}$$

   where $\hat{u}_0 \in L^2(\mathbb{R})$. Check that $u_\lambda(t) = e^{-a\partial_x t}u_\lambda(0)$. Show the a priori bound

   $$\|u_\lambda(t)\|_{L^2(\mathbb{R})} \leq e^{a\lambda t}\|u_\lambda(0)\|_{L^2(\mathbb{R})}.$$

5. Show there exists $u \in L^\infty(0, T; L^2(\mathbb{R}))$ such that

   $$\lim_{\lambda\to\infty} u_\lambda = u \text{ in } L^\infty(0, T; L^2(\mathbb{R})).$$

6. Generalize to the problem

   $$\begin{cases} \partial_t u = \partial_{xx} u, & t > 0, \quad x \in \mathbb{R}, \\ u_0 \in L^2(\mathbb{R}), & x \in \mathbb{R}. \end{cases}$$

Exercise 5: Convergence in weighted norm

The following example shows that the choice of the norm matters for the numerical analysis, specially if one takes weighted norms.

Take two positive parameters $a, \sigma > 0$. Let u be the solution of the transport equation with a source

$$\begin{cases} \partial_t u + a\partial_x u = \sigma u, & x \in \mathbb{R}, \quad t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

We will systematically assume that all solutions u tend sufficiently fast to zero at infinity.

1. Give the exact solution in function of the initial data.

   Hint: one can constructs a formula for the solution under the form $u(t, x) = \varphi(t)u_0(x - at)$.

2. Show that $\|u(t)\|_{L^2(\mathbb{R})} = e^{\sigma t}\|u_0\|_{L^2(\mathbb{R})}$.
3. Determine the symbol of the operator and determine its stability.
4. Consider the weighted norm

$$\|u\| = \|u\sqrt{w}\|_{L^2(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} u(x)^2 w(x)dx}$$

where $w > 0$ is the weight.

Find w such that

$$\|u(t)\| = \|u_0\|, \quad \forall t > 0,$$

for all possible u_0.

Hint: by derivation of the criterion, one can try to construct an equation satisfied by w and then solve the equation.

5. Consider the upwind scheme under the form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = \sigma u_{j-1}^n$$

where the source is also upwinded. The numerical solution at time step $t_n = n\Delta t$ is $u_h^n = \left(u_j^n\right)_{j\in\mathbb{Z}}$ and its discrete norm is

$$\|u_h^n\| = \sqrt{\sum_j |u_j^n|^2 w_j \Delta x}, \qquad w_j = e^{-\frac{2\sigma}{a} j\Delta x}.$$

Assume the CFL condition $\nu = a\frac{\Delta t}{\Delta x} \leq 1$. Show the inequality

$$\|u^{n+1}\| \leq Q(\sigma)\|u^n\|, \quad Q(\sigma) = 1 - \nu + (\nu + \sigma\Delta t)\, e^{-\frac{\sigma}{a}\Delta x}.$$

Indication: show that

$$\sum_j |u_j^{n+1}|^2 w_j = Q(\sigma)^2 \sum_j |u_j^n|^2 w_j - (1-\nu)(\nu + \sigma\Delta t)\sum_j \left(u_j^n\sqrt{w_{j+1}} - u_{j-1}^n\sqrt{w_j}\right)^2.$$

6. Show that $Q'(\sigma) \leq 0$ and prove the uniform stability (in the weighted discrete norm) of the upwind scheme.
7. Prove that the scheme is convergent (eventually under the simplified assumption that the support of the functions u_0 and u is compact in space).