Méthodes numériques pour les EDP instationnaires

TD 2: jeudi 22.09.2022
Stability and Fourier analysis

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1 Modified equations

Q1: Determine the modified equation of the Lax-Wendroff sheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{a^2\Delta t}{2}\frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{\Delta x^2} = 0\,.$$

Q2: Determine the modified equation of the Beam-Warming scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{3u_j^n - 4u_{j-1}^n + u_{j-2}^n}{2\Delta x} - \frac{a^2\Delta t}{2}\frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{\Delta x^2} = 0\,.$$

2 Stability and consistency

Q1: Show that the Beam-Warming scheme is L^2-stable for $\nu \leq 2$ and second order in time and space.

Q2: Show that the Fourier symbol of Lax-Friedrichs scheme is not consistent (for small Δt).

Q3: Study the 3-points scheme for diffusion

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \quad n \in \mathbb{N},\ j \in \mathbb{Z}.$$

Q4: Study the 3-points scheme (à la Dufort-Frankl) for diffusion

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^{n+1} + u_{j-1}^n}{\Delta x^2}, \quad n \in \mathbb{N},\ j \in \mathbb{Z}.$$

3 Maximum principle for advection

Show that a second order (space-time) linear explicit scheme cannot preserve the maximum principle for all CFL number.

Hint: Show that second order polynomials are exactly evolved with such a scheme.

4 Functional analysis in Fourier

This exercise can be considered as a pedestrian introduction to a much broader topic, see for example the Chapter 7 on the Hille-Yosida theorem of the book *Functional Analysis, Sobolev Spaces and Partial Differential Equations* by H. Brezis. The aim is to give a meaning to the representation formula $u(t) = e^{At}u_0$ where A is an unbounded operator. Typically $A = -a\partial_x$ or $A = \partial_{xx}$. The issue is that

$$e^{At} = \sum_{n\geq 0} A^n t^n$$

is meaningless for unbounded operators. One needs to construct such an object by an approximation procedure.

Q1: We consider the Fourier decomposition $v(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{v}(\theta) e^{i\theta x} d\theta$ and the truncation in Fourier space

$$v_\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} \hat{v}(\theta) e^{i\theta x} d\theta, \quad \lambda > 0.$$

Show that $\lim_{\lambda\to\infty} v_\lambda = v$ for $v \in L^2(\mathbb{R})$.
Measure the difference in $L^2(\mathbb{R})$ for $v \in H^s(\mathbb{R})$, $s \geq 1$.

Q2: Define the space

$$X_\lambda = \left\{ v \in L^2(\mathbb{R}),\ \widehat{v}(\theta) = 0 \text{ for } |\theta| > \lambda \right\}.$$

Show that

$$\|a\partial_x\|_{\mathcal{L}(X_\lambda)} \leq a\lambda.$$

Q3: Give a meaning in $\mathcal{L}(X_\lambda)$ to the series

$$e^{-a\partial_x t} = \sum_{n\geq 0} \frac{1}{n!}(-a\partial_x t)^n.$$

Q4: Consider the problem ($a > 0$)

$$\begin{cases} \partial_t u_\lambda + a\partial_x u_\lambda = 0, & t > 0, \quad x \in \mathbb{R}, \\ u_\lambda(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} \hat{u}_0(\theta) e^{i\theta x} d\theta \end{cases}$$

where $\hat{u}_0 \in L^2(\mathbb{R})$. What does mean $u_\lambda(t) = e^{-a\partial_x t} u_\lambda(0)$? Show the a priori bound

$$\|u_\lambda(t)\|_{L^2(\mathbb{R})} \leq e^{a\lambda t} \|u_\lambda(0)\|_{L^2(\mathbb{R})}.$$

Q5: Show there exists $u \in L^\infty(0, T; L^2(\mathbb{R}))$ such that

$$\lim_{\lambda\to\infty} u_\lambda = u \text{ in } L^\infty(0, T; L^2(\mathbb{R})).$$

Q6: Generalize to the problem

$$\begin{cases} \partial_t u = \partial_{xx} u, \quad t > 0, \quad x \in \mathbb{R}, \\ u_0 \in L^2(\mathbb{R}), \quad x \in \mathbb{R}. \end{cases}$$

5 Weighted norm

The following example shows that the choice of the norm matters for the numerical analysis, specially if one takes weighted norms.

Take two positive parameters $a, \sigma > 0$. Let u be the solution of the transport equation with a source

$$\begin{cases} \partial_t u + a\partial_x u = \sigma u, & x \in \mathbb{R}, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

We will systematically assume that all solutions u tend sufficiently fast to zero at infinity.

Q1: Give the exact solution in function of the initial data.

Hint: one can constructs a formula for the solution under the form $u(t,x) = \varphi(t)u_0(x - at)$.

Q2: Show that $\|u(t)\|_{L^2(\mathbb{R})} = e^{\sigma t}\|u_0\|_{L^2(\mathbb{R})}$.

Q3: Determine the symbol of the operator and determine its stability.

Q4: Consider the weighted norm

$$\|u\| = \|u\sqrt{w}\|_{L^2(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} u(x)^2 w(x) dx}$$

where $w > 0$ is the weight.

Find w such that

$$\|u(t)\| = \|u_0\|, \quad \forall t > 0,$$

for all possible u_0.

Indication: by derivation of the criterion, one can try to construct an equation satisfied by w and then solve the equation.

Q5: Consider the upwind scheme under the form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = \sigma u_{j-1}^n$$

where the source is also upwinded. The numerical solution at time step $t_n = n\Delta t$ is $u_h^n = \left(u_j^n\right)_{j\in\mathbb{Z}}$ and its discrete norm is

$$\|u_h^n\| = \sqrt{\sum_j |u_j^n|^2 w_j \Delta x}, \qquad w_j = e^{-\frac{2\sigma}{a} j\Delta x}.$$

Assume the CFL condition $\nu = a\frac{\Delta t}{\Delta x} \leq 1$. Show the inequality

$$\|u^{n+1}\| \leq Q(\sigma)\|u^n\|, \quad Q(\sigma) = 1 - \nu + (\nu + \sigma\Delta t)\, e^{-\frac{\sigma}{a}\Delta x}.$$

Indication: show that

$$\sum_j |u_j^{n+1}|^2 w_j = Q(\sigma)^2 \sum_j |u_j^n|^2 w_j - (1-\nu)(\nu + \sigma\Delta t) \sum_j \left(u_j^n \sqrt{w_{j+1}} - u_{j-1}^n \sqrt{w_j}\right)^2.$$

Q6: Show that $Q'(\sigma) \leq 0$ and prove the uniform stability (in the weighted discrete norm) of the upwind scheme.

Q7: Prove that the scheme is convergent (eventually under the simplified assumption that the support of the functions u_0 and u is compact in space).