Méthodes numériques pour les EDP instationnaires

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Transport equation with constant coefficients

For a given $a \in \mathbb{R}$, we consider the following linear transport equation in one dimension :

$$\begin{cases} \partial_t \bar{u} + a\ \partial_x \bar{u} = 0\,, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}_*^+, \\ \bar{u}(x,0) = u_0(x)\,, & \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

with $u_0 \in L^\infty(\mathbb{R})$. Without loss of generality, we assume that $a > 0$. We refer to the chapter 2, subsection 2.2.1, for the continuous framework of this equation. Here we focus on finding u a discrete approximation of $\bar{u}$ thanks to discrete schemes. As in chapter 3, we introduce a discretization of the domain using a regular mesh : $(x_j, t_n) = (j\Delta x, n\Delta t),\ \forall j \in \mathbb{Z},\ \forall n \in \mathbb{N}$, where Δx, respectively Δt, denotes the space step, respectively the time step. We also denote u_j^n the approximation of $\bar{u}(x_j, t_n)$.

Definition: A scheme is L^∞ stable if we can prove the estimate

$$\sup_j \left|u_j^{n+1}\right| \leq \sup_j \left|u_j^n\right|.$$

Definition: A scheme is L^2 stable if we can prove the estimate

$$\sum_j \left|u_j^{n+1}\right|^2 \leq \sum_j \left|u_j^n\right|^2.$$

1 Lax-Wendroff scheme

We first focus on the *Lax-Wendroff* scheme :

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\ \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{a^2 \Delta t}{2}\ \frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{\Delta x^2} \ = \ 0\ . \tag{2}$$

1. **Truncation error**

   The exact solution $\bar{u}$ of (1) is generally not a solution of the scheme (2). The truncation error estimates the difference. Let us assume that the solution of (1) is such that $\bar{u} \in C^3(\mathbb{R} \times \mathbb{R}^+)$.

   (a) Prove that, for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, $\partial_{tt}\bar{u} = a^2\ \partial_{xx}\bar{u}$.

   (b) Compute the Taylor expansions ("développements limités avec reste de Taylor-Lagrange") at a convenient order of $\bar{u}(x_j, t_{n+1})$, $\bar{u}(x_{j+1}, t_n)$, and $\bar{u}(x_{j-1}, t_n)$ at the point (x_j, t_n).

   (c) Assuming that enough partial derivatives of $\bar{u}$ are bounded in L^∞ norm by some constant $C \in \mathbb{R}_*^+$, prove that the absolute value of the truncation error of the Lax-Wendroff scheme is second order both in time and space.

2. **L^∞ stability**

   (a) Show that, for any non-negative values α, β, γ such that $\alpha + \beta + \gamma = 1$, then

   $$\forall x, y, z \in \mathbb{R}, \min(x,y,z) \leq \alpha x + \beta y + \gamma z \leq \max(x,y,z).$$

   (b) Using (2), find α, β, γ such that $u_j^{n+1} = \alpha u_j^n + \beta u_{j+1}^n + \gamma u_{j-1}^n$.

   (c) Provide a necessary and sufficient condition on Δt, Δx and a ensuring the non-negativity of the coefficients α, β, γ found at the previous question. Deduce the L^∞ stability domain of the scheme.

3. **L^2 stability**

   (a) Show that

   $$\sum_j \left|u_j^{n+1}\right|^2 = \sum_j \left|u_j^n\right|^2 - \frac{\nu^2(1-\nu^2)}{4} \sum_j \left|w_{j+1}^n - w_j^n\right|^2,$$

   where $\nu = \dfrac{a\Delta t}{\Delta x}$ and $w_j^n = u_j^n - u_{j-1}^n$.

   (b) Deduce the condition under which the scheme is L^2 stable.

2 Schemes overview

- *Centered explicit scheme*

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \;=\; 0. \tag{3}$$

- *Centered implicit scheme*

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + a\,\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} \;=\; 0. \tag{4}$$

- *Upwind scheme*

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t} + a\,\dfrac{u_j^n - u_{j-1}^n}{\Delta x} \;=\; 0 \text{ , if } a>0, \\ \dfrac{u_j^{n+1}-u_j^n}{\Delta t} + a\,\dfrac{u_{j+1}^n - u_j^n}{\Delta x} \;=\; 0 \text{ , if } a<0. \end{cases} \tag{5}$$

- *Lax-Friedrichs*

$$\frac{2u_j^{n+1}-u_{j+1}^n - u_{j-1}^n}{2\Delta t} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \;=\; 0. \tag{6}$$

- *Beam-Warming* (if $a > 0$)

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + a\,\frac{3u_j^n - 4u_{j-1}^n + u_{j-2}^n}{2\Delta x} - \frac{a^2\Delta t}{2}\,\frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{\Delta x^2} \;=\; 0. \tag{7}$$

1. We assume that u_0 is a periodic function. Unlike the other schemes, the *centered implicit* scheme does not allow, for a given space index j and a given time index n, to express explicitly u_j^{n+1} in function of the $(u_k^n)_k$. A linear system has to be solved. Construct the matrix of the linear system, prove it is invertible. Show the L^2 stability unconditionally (*Hint:* compute U^tAU).

   We sum up in the table below some properties of each scheme :

scheme	**stability**	**truncation error**
Lax-Wendroff	L^2 stable under CFL $\lvert a\rvert\Delta t \le \Delta x$ $\quad$ [L^∞ stable if $\lvert a\rvert\Delta t = \Delta x$]	$O\big((\Delta t)^2 + (\Delta x)^2\big)$
centered explicit	unstable	$O\big(\Delta t + (\Delta x)^2\big)$
centered implicit	unconditionally L^2 stable	$O\big(\Delta t + (\Delta x)^2\big)$
upwind	L^2 and L^∞ stable under CFL $\lvert a\rvert\Delta t \le \Delta x$	$O(\Delta t + \Delta x)$
Lax-Friedrichs	L^2 and L^∞ stable under CFL $\lvert a\rvert\Delta t \le \Delta x$	$O\left(\Delta t + \dfrac{(\Delta x)^2}{\Delta t}\right)$
Beam-Warming	L^2 stable under CFL $\lvert a\rvert\Delta t \le 2\Delta x$	$O\big((\Delta t)^2 + (\Delta x)^2\big)$

2. Do you see one advantage to use the *Beam-Warming* scheme?

3. For the following schemes: *Lax-Wendroff*, *upwind*, *Lax-Friedrichs* and *Beam-Warming*, show that if $a\Delta t = \Delta x$, the numerical solution u_j^n is equal to the analytical solution at the discretization point (x_j, t_n).

4. By using the same tools as the ones used for the *Lax-Wendroff* scheme in section one, for each scheme of the table above, check its stability properties and its truncation error.

5. Assuming $a > 0$, we introduce the third order scheme,

$$O3 = (1-\delta)LW + \delta BW \;\;,\;\; \delta = \frac{1+\nu}{3} \tag{8}$$

   where LW denotes the Lax-Wendroff scheme and BW denotes the Beam-Warming scheme. Check that this scheme is of order 3 in space and in time.